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Small embedding of an $S_3(2, 4, u)$ into an $S_\lambda(2, 4, u + w)$

Salvatore Milici [a], Giorgio Ragusa [a], Fulvio Zuanni [b]

[a] Dipartimento di Matematica e Informatica, Università di Catania, Catania, Italy

[b] Dipartimento di Ingegneria Elettrica e dell'Informazione, Università di L'Aquila, L'Aquila, Italy

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ABSTRACT

Let H be a subgraph of a graph G. An H-design $(U, \mathcal{C})$ of order u and index μ is embedded into a G-design $(V, \mathcal{B})$ of order v and index λ if $\mu \leq \lambda$, $U \subseteq V$ and there is an injective mapping $f : \mathcal{C} \to \mathcal{B}$ such that B is a subgraph of $f(B)$ for every $B \in \mathcal{C}$. The mapping f is called the embedding of $(U, \mathcal{C})$ into $(V, \mathcal{B})$. We determine, for every admissible value of u and λ, the minimum value of w (except 12 values of (u, λ)) such that every $S_3(2, 4, u)$ can be embedded into an $S_\lambda(2, 4, u + w)$. This result implies that we determine also the minimum value of w such that there exists an $S_\lambda(2, 4, u + w)$ which embeds an $E_2(u, 1)$, where E_2 is the graph with two parallel edges and without isolated vertices.

1. Introduction and definitions

Let G be a finite and simple graph. A G-design of order v and index λ is a pair $(V, \mathcal{C})$ where V is the vertex set of K_v (the complete graph on v vertices) and $\mathcal{C}$ is a collection of isomorphic copies of the graph G, called *blocks*, which partition the edges of λK_v (the complete multigraph on v vertices).

A K_4-design of order v and index λ is well known as a balanced incomplete block design of order v, index λ and block-size 4. We denote such a design by $S_\lambda(2, 4, v)$. Hanani [6] proved that an $S_\lambda(2, 4, v)$ exists if and only if

- $v \equiv 1, 4 \pmod{12}$ if $\lambda \equiv 1, 5 \pmod 6$;
- $v \equiv 1 \pmod 3$ if $\lambda \equiv 2, 4 \pmod 6$;
- $v \equiv 0, 1 \pmod 4$ if $\lambda \equiv 3 \pmod 6$;
- any $v \geq 4$ if $\lambda \equiv 0 \pmod 6$.

Definition 1.1. Let H be a subgraph of G, and let $U \subseteq V$. We say that an H-design $(U, \mathcal{C})$ of order u and index μ is embedded into a G-design $(V, \mathcal{B})$ of order $u + w$ and index λ, $\mu \leq \lambda$, if there is an injective mapping

$$f : \mathcal{C} \to \mathcal{B}$$

such that B is a subgraph of $f(B)$ for every $B \in \mathcal{C}$.

The mapping f is called the embedding of $(U, \mathcal{C})$ into $(V, \mathcal{B})$. When w attains the minimum possible value we say that f is a minimum embedding.

If $H = G$ and $\mu = \lambda$ then we obtain the usual embedding definition for G-designs.

When $\mu = \lambda = 1$, the (minimum) embedding of an H-design into a G-design has been studied for many pairs of graphs H and G with H a subgraph of G (see [3,10] for a survey).

When $\mu = 1$ and $\lambda > 1$ the minimum embedding has been studied by Milici [7] for $H = P_3$ and $G = K_3$, by Gionfriddo et al. [5,9] for $H = K_3$ and $G = K_4$, by Danziger et al. [4] for $H = P_4$ and $G = K_4$. Milici and Ragusa [8] have studied also the

E-mail addresses: milici@dmi.unict.it (S. Milici), gragusa@dmi.unict.it (G. Ragusa), fulvio.zuanni@univaq.it (F. Zuanni).

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Table 1
Minimum embedding of an $S_3(2,4,u)$ into an $S_\lambda(2,4,u+w)$.

$\lambda \pmod 6$, $\lambda \geq 3$	$u \pmod{12}$, $u \geq 4$	w
3	0, 1, 4, 5, 8, 9	0
2, 4	1, 4	0
	0, 9	1
	5,8 ($u \geq 17$ for $\lambda = 4, 8$)	2[a]
1, 5	1, 4	0
	0	1
	5	8[b]
	8	5[c]
	9 ($u \geq 21$ for $\lambda = 5$)	4
0	$\forall$	0
$\lambda = 4$	$u = 5, 8$	11, 14
$\lambda = 8$	$u = 5, 8$	5, 2
$\lambda = 5$	$u = 9$	≥ 7

[a] With possible exceptions for $\lambda = 4$ and $u = 29, 32, 41, 44, 53, 56, 65$.
[b] With possible exceptions for $\lambda = 5$ and $u = 29, 53$.
[c] With possible exceptions for $\lambda = 5$ and $u = 32, 44$.

maximum embedding for $H = H_3$ and $G = K_3$. In this paper we wish to consider the minimum embedding of an $S_3(2,4,u)$ into an $S_\lambda(2,4,u+w)$, $\lambda \geq 3$. In particular, we will prove the following result:

Main Theorem. *Let $u \equiv 0, 1 \pmod 4$ and $\lambda \geq 3$. Every $S_3(2,4,u)$ can be embedded into an $S_\lambda(2,4,u+w)$ of minimum order $u+w$ if and only if the conditions in Table 1 are satisfied.*

2. Preliminaries and necessary conditions

In this section we recall some useful definitions and results. For terms not defined in this paper or results not explicitly cited the reader is referred to [2] and its online updates.

A *pairwise balanced design PBD*(v, K) of order v with block sizes from K is a pair $(V, \mathcal{B})$, where V is a finite set of cardinality v and $\mathcal{B}$ is a family of subsets of V (*blocks*) such that $|B| \in K$ for every $B \in \mathcal{B}$ and every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$.

A 4-GDD is a triple $(V, \mathcal{G}, B)$, where V is a finite set, $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ is a partition of V into subsets, the elements of $\mathcal{G}$ are called *groups*, and $\mathcal{B}$ is a collection of isomorphic copies of K_4, called *blocks*, which partition the edges of $K_{g_1,g_2,\ldots,g_n}$ ($|G_i| = g_i$) on the vertex set V. If for $i = 1, 2, \ldots, t$, there are u_i groups of size g_i, we say that the 4-GDD is of type $g_1^{u_1} g_2^{u_2}, \ldots, g_t^{u_t}$.

Let $\mathcal{B}$ be the block set of a design. A *parallel class* or *resolution class* is a collection of blocks which partition the point-set of the design. A design is *resolvable* if $\mathcal{B}$ can be partitioned into parallel classes.

We recall the existence of some 4-GDD and $PBD(v, K)$ we need in the following.

Lemma 2.1 ([2]). *There exists a* 4*-GDD of type*

- $u^1 1^t$ *for each* $u \equiv 4, 10 \pmod{12}$, $t \equiv 0, 9 \pmod{12}$, $t \geq 2u+1$;
- $u^1 1^t$ *for each* $u \equiv 1, 7 \pmod{12}$, $t \equiv 0, 3 \pmod{12}$, $t \geq 2u+1$.

Lemma 2.2 ([2]). *There exists a $PBD(v, \{4,5\})$ for each $v \equiv 0, 1 \pmod 4$, $v \neq 8, 9, 12$.*

Lemma 2.3 ([1]). *A $PBD(v, \{4, 7^*\})$, that is a pairwise balanced design on v point with blocks of sizes* 4 *and exactly one block of size* 7 *exists if and only if $v \equiv 7, 10 \pmod{12}$, $v \neq 10, 19$.*

Lemma 2.4. *Let $u \equiv 0, 1 \pmod 4$ and $0 \leq w < 2u+1$, $\lambda > 3$. If there exists an $S_\lambda(2,4,u+w)$ which embeds an $S_3(2,4,u)$ then*

$$3\lambda w^2 - \lambda w(2u+3) + (\lambda - 3)u(u-1) \geq 0.$$

Proof. Let $u \equiv 0, 1 \pmod 4$, $W = \{a_i : i \in \mathbb{Z}_w\}$ and $V = \mathbb{Z}_u \cup W$. Suppose we embed an $S_3(2,4,u)(\mathbb{Z}_u, \mathcal{C})$ into an $S_\lambda(2,4,u+w)(V, \mathcal{B})$. Simple counting arguments show that $|\mathcal{C}| = \frac{u(u-1)}{4}$, $|\mathcal{B}| = \frac{\lambda(u+w)(u+w-1)}{12}$ and every vertex of V occurs in $\lambda\frac{u+w-1}{3}$ blocks of $\mathcal{B}$. Since $w < 2u+1$, the vertices of W occur in at least $\lambda\left[\frac{w(u+w-1)}{3} - \frac{w(w-1)}{2}\right]$ blocks of $\mathcal{B} \setminus \mathcal{C}$. Then necessarily we must have

$$\lambda\left[\frac{w(u+w-1)}{3} - \frac{w(w-1)}{2}\right] \leq \lambda\frac{(u+w)(u+w-1)}{12} - \frac{u(u-1)}{4}$$

which is equivalent to

$$3\lambda w^2 - \lambda w(2u+3) + (\lambda-3)u(u-1) \geq 0. \quad \square$$

Lemma 2.5. *Let $u \equiv 0, 1 \pmod 4$, $\lambda \geq 3$ and $w \geq \frac{u-1}{2}$. If there exists an $S_\lambda(2, 4, u+w)$ which embeds an $S_3(2, 4, u)$ then*

$$\lambda w^2 - \lambda w(2u+1) + 3(\lambda-3)u(u-1) \geq 0.$$

Proof. Let $u \equiv 0, 1 \pmod 4$ and $W = \{a_i : i \in \mathbb{Z}_w\}$. Suppose we embed an $S_3(2, 4, u)(\mathbb{Z}_u, \mathcal{C})$ into an $S_\lambda(2, 4, u+w)(\mathbb{Z}_u \cup W, \mathcal{B})$. Simple counting arguments show that $|\mathcal{C}| = \frac{u(u-1)}{4}$, $|\mathcal{B}| = \frac{\lambda\,(u+w)(u+w-1)}{12}$ and every vertex of $\mathbb{Z}_u$ occurs in $u-1$ blocks of $\mathcal{C}$ and

$$\lambda\frac{u+w-1}{3} - (u-1) = \frac{(\lambda-3)(u-1)+\lambda w}{3}$$

blocks of $\mathcal{B} \setminus \mathcal{C}$. Since $w \geq \frac{u-1}{2}$, the vertices of $\mathbb{Z}_u$ occur in at least

$$\frac{(\lambda-3)u(u-1)+\lambda uw}{3} - \frac{(\lambda-3)u(u-1)}{2} = \frac{2\lambda uw - (\lambda-3)u(u-1)}{6}$$

blocks of $\mathcal{B} \setminus \mathcal{C}$. Then necessarily we must have

$$\frac{2\lambda uw - (\lambda-3)u(u-1)}{6} \leq \lambda\frac{(u+w)(u+w-1)}{12} - \frac{u(u-1)}{4}$$

which is equivalent to

$$\lambda w^2 - \lambda w(2u+1) + 3(\lambda-3)u(u-1) \geq 0. \quad \square$$

Applying Lemmas 2.4 and 2.5 with $u = 5, 8, 9$ and the spectrum of $S_\lambda(2, 4, u)$, we obtain the following

Corollary 2.6. *If there exists an*

- $S_\lambda(2, 4, 5+w)$ *which embeds an* $S_3(2, 4, 5)$, *then* $\lambda \geq 10$ *for* $w = 2$, $w \geq 11$ *for* $\lambda = 4$ *and* $w \geq 5$ *for* $\lambda = 8$;
- $S_\lambda(2, 4, 8+w)$ *which embeds an* $S_3(2, 4, 8)$, *then* $\lambda \geq 6$ *for* $w = 2$ *and* $w \geq 14$ *for* $\lambda = 4$;
- $S_\lambda(2, 4, 9+w)$ *which embeds an* $S_3(2, 4, 9)$, *then* $\lambda \geq 6$ *for* $w = 4$ *and* $w \geq 7$ *for* $\lambda = 5$.

3. Proof of Main Theorem

The necessary part of the Main Theorem follows easily from the necessary and sufficient conditions for the existence of an $S_3(2, 4, u)$ and an $S_\lambda(2, 4, u+w)$ and from Corollary 2.6. It is easy to see that the sufficiency of Main Theorem for $\lambda = 3, 4, 5, 6, 7, 8, 10$ implies its sufficiency for every $\lambda \geq 3$, with $\lambda = a + 6k$, $a = 0, 1, 2, 3, 4, 5$. The minimum embedding is obtained:

- for $a = 0, 1, 2$ and $k \geq 1$, by pasting the blocks of an $S_{a+6}(2, 4, u+w)$ which embeds the given $S_3(2, 4, u)$ to the blocks of an $S_{6(k-1)}(2, 4, u+w)$
- for $a = 3$ and $k \geq 1$, by pasting the blocks of the given $S_3(2, 4, u)$ to the blocks of an $S_{6k}(2, 4, u)$
- for $a = 4$, $u \neq 5$ and $k \geq 1$, by pasting the blocks of an $S_8(2, 4, u+w)$ which embeds the given $S_3(2, 4, u)$ to the blocks of an $S_{6k-4}(2, 4, u+w)$,
- for $a = 4$, $u = 5$ and $k \geq 2$, by pasting the blocks of an $S_{10}(2, 4, 5+w)$ which embeds the given $S_3(2, 4, 5)$ to the blocks of an $S_{6k-6}(2, 4, 5+w)$
- for $a = 5$ and $k \geq 1$, by pasting the blocks of an $S_7(2, 4, u+w)$ which embeds the given $S_3(2, 4, u)$ to the blocks of an $S_{6k-2}(2, 4, u+w)$.

3.1. $\lambda = 4$

For $u \equiv 1, 4 \pmod{12}$ the proof of the Main Theorem follows by pasting an $S(2, 4, u)$ to the given $S_3(2, 4, u)$. For $u = 5$ and $u = 8$ the proof follows from Corollary 2.6 and cases 6, 9 in the Appendix.

Theorem 3.1. *If $u \equiv 0, 9 \pmod{12}$, $u \geq 9$ then every $S_3(2, 4, u)$ can be embedded into an $S_4(2, 4, u+1)$.*

Proof. Let $(\mathbb{Z}_u, \mathcal{C})$ be an $S_3(2, 4, u)$. Construct a 4-GDD of type $4^1 1^u$ on $\mathbb{Z}_u \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$ having $\{\infty_0, \infty_1, \infty_2, \infty_3\}$ as group of size 4 and $\mathcal{B}$ as the block-set. Let $\bar{\mathcal{B}}$ be the block-set obtained from $\mathcal{B}$ by replacing,for each $i \in \mathbb{Z}_4$, ∞_i with ∞. It is easy to check that $(\mathbb{Z}_u \cup \{\infty\}, \mathcal{C} \cup \bar{\mathcal{B}})$ is the required design. $\square$

Theorem 3.2. *If $u \equiv 5, 8 \pmod{12}$, $u \geq 17$ and $u \neq 29, 32, 41, 44, 53, 56, 65$, then every $S_3(2, 4, u)$ can be embedded into an $S_4(2, 4, u+2)$.*

Proof. For $u = 17, 20$, see cases 7, 8 in the Appendix. For $u \geq 68$, write $u = x + 17 + 12t$, $t \geq 4$ and $x = 0, 3$. Now let $X = \{a_0, a_1, \ldots, a_{16}\}$ (or $X = \{a_0, a_1, \ldots, a_{19}\}$ for $x = 3$), $U = \mathbb{Z}_{u-17} \cup X$ (or $U = \mathbb{Z}_{u-20} \cup X$ for $x = 3$) and $(U, \mathcal{D})$

be an $S_3(2,4,u)$. Construct a 4-GDD of type $25^1 1^{u-17}$ (or of type $28^1 1^{u-20}$ for $x = 3$) on $U \cup \{\infty_i, \overline{\infty_i} : i \in \mathbb{Z}_4\}$ having $X \cup \{\infty_i, \overline{\infty_i} : i \in \mathbb{Z}_4\}$ as group of size 25 (or 28 for $x = 3$) and $\mathcal{B}$ as the block-set. Let $\bar{\mathcal{B}}$ be the block-set obtained from $\mathcal{B}$ by replacing, for each $i \in \mathbb{Z}_4$, ∞_i with ∞_1 and $\overline{\infty_i}$ with ∞_2. Place on $X_1 = X \cup \{\infty_1, \infty_2\}$ an $S_4(2,4,19)(X_1, \mathcal{B}_1)$ which embeds an $S_3(2,4,17)(X, \mathcal{C}_1)$ on X (see cases 7, 8 in the Appendix). It is easy to check that $(U \cup \{\infty_1, \infty_2\}, \mathcal{D} \cup \bar{\mathcal{B}} \cup (\mathcal{B}_1 \setminus \mathcal{C}_1))$ is the required design. □

3.2. $\lambda = 5$

For $u \equiv 1, 4 \pmod{12}$ the proof of the Main Theorem follows by pasting an $S_2(2,4,u)$ to the given $S_3(2,4,u)$ and for $u \equiv 0 \pmod{12}$, $u \geq 12$, by pasting an $S(2,4,u+1)$ to an $S_4(2,4,u+1)$ which embeds the given $S_3(2,4,u)$. So we suppose $u \equiv 5, 8, 9 \pmod{12}$. For $u = 9$ the proof follows from Corollary 2.6.

Theorem 3.3. *If $u \equiv 5 \pmod{12}$, $u \neq 29, 53$, then every $S_3(2,4,u)$ can be embedded into an $S_5(2,4,u+8)$.*

Proof. For $u = 5, 17, 41$, see cases 10, 13, 15 in the Appendix. For $u \geq 65$ write $u = 5 + 12t$, $t \geq 5$. Now let $X = \{a_0, a_1, a_2, a_3, a_4\}$, $U = \mathbb{Z}_{u-5} \cup X$ and $(U, \mathcal{D})$ be an $S_3(2,4,u)$. Construct a 4-GDD of type $25^1 1^{u-5}$ (see Lemma 2.1) on $\mathbb{Z}_{u-5} \cup \{a_{ij} : (i,j) \in \mathbb{Z}_5 \times \mathbb{Z}_2\} \cup \{b_{ij} : (i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_5\}$ and a 4-GDD of type $25^1 1^{u-5}$ on $\mathbb{Z}_{u-5} \cup \{\infty_{ij} : (i,j) \in \mathbb{Z}_5 \times \mathbb{Z}_5\}$. For each $j \in \mathbb{Z}_2$, replace a_{ij} with a_i, for each $k \in \mathbb{Z}_5$, replace ∞_{ik} with ∞_i and b_{ik} with b_i and denote by $\bar{\mathcal{B}}$ the block-set so obtained. On $\{a_0, a_1, a_2, a_3, a_4\} \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\} \cup \{b_0, b_1, b_2\}$, place an $S_5(2,4,13)(V_1, \mathcal{B}_1)$ which embeds an $S_3(2,4,5)(X, \mathcal{C}_1)$ on $\{a_0, a_1, a_2, a_3, a_4\}$ (see case 10 in the Appendix). Let $V = U \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\} \cup \{b_0, b_1, b_2\}$, $\mathcal{C} = \mathcal{B}_1 \setminus \mathcal{C}_1$, $\mathcal{B} = \mathcal{D} \cup \bar{\mathcal{B}} \cup \mathcal{C}$. It is easy to check that (V, B) is the required design. □

Theorem 3.4. *If $u \equiv 8 \pmod{12}$, $u \neq 32, 44$, then every $S_3(2,4,u)$ can be embedded into an $S_5(2,4,u+5)$.*

Proof. For $u = 8, 20$, see cases 11 and 14 in the Appendix. For $u \geq 56$ write $u = 8 + 12t$, $t \geq 4$. Now let $X = \{a_i, i \in \mathbb{Z}_8\}$, $U = \mathbb{Z}_{u-8} \cup X$ and $(U, \mathcal{D})$ be an $S_3(2,4,u)$. Construct a 4-GDD of type $19^1 1^{u-8}$ (see Lemma 2.1) on $\mathbb{Z}_{u-8} \cup \{a_{ij} : (i,j) \in \mathbb{Z}_8 \times \mathbb{Z}_2\} \cup \{\infty_{4j} : j \in \mathbb{Z}_3\}$ and a 4-GDD of type $22^1 1^{u-8}$ on $\mathbb{Z}_{u-8} \cup \{\infty_{ij} : (i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_5\} \cup \{\infty_{4j} : j = 3, 4\}$. For each $j \in \mathbb{Z}_2$, replace a_{ij} with a_i, for each $k \in \mathbb{Z}_5$, replace ∞_{ik} with ∞_i and denote by $\bar{\mathcal{B}}$ the block-set so obtained. On $X \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$, place an $S_5(2,4,13)(V_1, \mathcal{B}_1)$ which embeds an $S_3(2,4,8)(X, \mathcal{C}_1)$ on X. Let $V = U \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$, $\mathcal{C} = \mathcal{B}_1 \setminus \mathcal{C}_1$, $\mathcal{B} = \mathcal{D} \cup \bar{\mathcal{B}} \cup \mathcal{C}$. It is easy to check that (V, B) is the required design. □

Theorem 3.5. *If $u \equiv 9 \pmod{12}$, $u \geq 21$ then every $S_3(2,4,u)$ can be embedded into an $S_5(2,4,u+4)$.*

Proof. Let $(\mathbb{Z}_u, \mathcal{D})$ be an $S_3(2,4,u)$. For $u = 21$, see case 12 in the Appendix. For $u \geq 33$ write $u = 9 + 12t$, $t \geq 2$. Take a 4-GDD of type $16^1 1^u$ (see Lemma 2.1) on $\mathbb{Z}_u \cup \{\infty_{ij} : (i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_4\}$ having $G = \{\infty_{ij} : (i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_4\}$ as group of size 16 and $\mathcal{B}_1$ as the block-set. Let $\bar{\mathcal{B}}$ be the block-set obtained from $\mathcal{B}_1$ by replacing, for each $j \in \mathbb{Z}_4$, $\infty_{i,j}$ with ∞_i. Put in $\mathcal{C}$ the blocks of an $S_4(2,4,4)$ on $\{\infty_0, \infty_1, \infty_2, \infty_3\}$ and the blocks of an $S(2,4,13+12t)$ on $V = U \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$. Let $\mathcal{B} = \mathcal{D} \cup \bar{\mathcal{B}} \cup \mathcal{C}$. It is easy to check that $(V, \mathcal{B})$ is the required design. □

3.3. $\lambda = 6$

The proof of the Main Theorem follows by doubling the solution for $\lambda = 3$. The following result will be used in this paper.

Theorem 3.6. *If $u \equiv 5, 8 \pmod{12}$, then every $S_3(2,4,u)$ can be embedded into an $S_6(2,4,u+1)$.*

Proof. For $u = 5, 8, 17$, see cases 16, 18 and 19 in the Appendix. For $u \geq 20$, write $u = x + 5 + 12t$, $t \geq 2$ and $x = 0, 3$. Let $(U, \mathcal{D})$ be an $S_3(2,4,u)$ where $U = \mathbb{Z}_{u-5} \cup \{a_0, a_1, a_2, a_3, a_4\}$. Construct a 4-GDD of type $7^1 1^u$ on $U \cup \{\infty_0, \infty_1\}$ having $\{a_0, a_1, a_2, a_3, a_4\} \cup \{\infty_0, \infty_1\}$ as the group of size 7. Replace, for each $i \in \mathbb{Z}_2$, ∞_i with ∞ and repeat the blocks so obtained three times. Develop (mod 5) the base blocks $\{\infty, a_0, a_1, a_2\}$, $\{\infty, a_0, a_1, a_3\}$. The result is an $S_6(2,4,u+1)$ on $V = U \cup \{\infty\}$ which embeds the $S_3(2,4,u)(U, \mathcal{D})$. □

3.4. $\lambda = 7$

For $u \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ and for $u \neq 9, 29, 32, 44, 53$ the proof of the Main Theorem follows by pasting an $S_2(2,4,u+w)$ to an $S_5(2,4,u+w)$ which embeds the given $S_3(2,4,u)$. For $u = 9, 29, 32, 44, 53$, see cases 22, 23, 24, 26, 25 in the Appendix.

3.5. $\lambda = 8$

For $u \equiv 0, 1, 4, 9 \pmod{12}$ the proof of the Main Theorem follows by doubling the solution for $\lambda = 4$. So we suppose $u \equiv 5, 8 \pmod{12}$. For $u = 5$ the proof follows from Corollary 2.6, by embedding the given $S_3(2,4,5)$ into an $S_6(2,4,10)$ (see case 17 in the Appendix) and by adding the blocks of an $S_2(2,4,10)$.

Theorem 3.7. *If $u \equiv 5 \pmod{12}$, $u \geq 17$ then every $S_3(2,4,u)$ can be embedded into an $S_8(2,4,u+2)$.*

Proof. Let $U = \mathbb{Z}_{u-7} \cup \{a_i, i \in \mathbb{Z}_7\}$. Embed an $S_3(2,4,u)$ on U into an $S_6(2,4,u+1)$ on $U \cup \{\infty_0\}$. Construct on $U \cup \{c_0, c_1, c_2, c_3\} \cup \{\infty_0\}$ a $PBD(10+12t, \{4, 7^*\})$ having $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$ as a block of size 7 and $\{c_0, c_1, c_2, c_3\}$ as a block of size 4. Replace, for each $i \in \mathbb{Z}_4$, c_i with ∞ and repeat the blocks so obtained twice, after removing the block of size 7 and the block $\{c_0, c_1, c_2, c_3\}$. Place on $\{a_i, i \in \mathbb{Z}_7\}$ an $S_2(2,4,7)$. The result is an $S_8(2,4,u+2)$ on $V = U \cup \{\infty_0, \infty\}$ which embeds an $S_3(2,4,u)$ on U. □

Theorem 3.8. *If* $u \equiv 8 \pmod{12}$, $u \geq 8$ *then every* $S_3(2,4,u)$ *can be embedded into an* $S_8(2,4,u+2)$.

Proof. For $u = 8$, see case 27 in the Appendix. For $u \geq 20$ embed an $S_3(2,4,u)$ on $\mathbb{Z}_u$ into an $S_6(2,4,u+1)$ on $\mathbb{Z}_u \cup \{\infty_0\}$. Construct a 4-GDD of type $4^1 1^{u+1}$ on $\mathbb{Z}_u \cup \{\infty_0\} \cup \{a_0, a_1, a_2, a_3\}$ having $\{a_0, a_1, a_2, a_3\}$ as the group of size 4. Replace, for each $i \in \mathbb{Z}_4$, a_i with ∞ and repeat the blocks so obtained twice. The result is an $S_8(2,4,u+2)$ on $V = \mathbb{Z}_u \cup \{\infty_0, \infty\}$ which embeds an $S_3(2,4,u)$ on $\mathbb{Z}_u$. □

3.6. $\lambda = 10$

For $u = 5$, see case 28 in the Appendix. For $u \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ and $u \neq 5$ the proof of the Main Theorem follows by pasting an $S_2(2,4,u+w)$ to an $S_8(2,4,u+w)$ which embeds an $S_3(2,4,u)$.

4. Applications for other designs

In this section,we shall use the Main Theorem to give new results on E_2-designs. Let E_2 be the graph $[a, b; c, d]$ having vertices $\{a, b, c, d\}$ and edges $\{a, b\}$, $\{c, d\}$. An E_2-design of order u and index 1, $E_2(u, 1)$, exists if and only if $u \equiv 0, 1 \pmod 4$.

Lemma 4.1. *Let* $u \equiv 0, 1 \pmod 4$. *If there exists an* $S_\lambda(2,4,u+w)$ *which embeds an* $E_2(u,1)$ *then* $\lambda \geq 3$.

Proof. Let $u \equiv 0, 1 \pmod 4$. Suppose we embed an $E_2(u,1)(U, \mathcal{C})$ into an $S_\lambda(2,4,u+w)(V, \mathcal{B})$. Counting the number of edges of λK_u not covered by blocks of $\mathcal{C}$ we obtain $\lambda \frac{u(u-1)}{2} \geq 6 \frac{u(u-1)}{4}$ and hence $\lambda \geq 3$.

Lemma 4.2. *If* $u \equiv 0, 1 \pmod 4$, $u \geq 4$ *then there is an* $S_3(2,4,u)$ *which embeds an* $E_2(u,1)$.

Proof. For $u = 4, 5, 8, 9, 12$, see cases 1,2,3,4 and 5 in the Appendix. For $u \geq 13$, take a $PBD(u, \{4, 5\})$ (see Lemma 2.2) and place on each block an $S_3(2,4,k)$ which embeds an $E_2(k,1)$, with $k = 4, 5$. □

Now using the results of the Main Theorem and Lemma 4.2 we obtain the following new results for an $E_2(u,1)$.

Theorem 4.3. *Let* $u \equiv 0, 1 \pmod 4$ *and* $\lambda \geq 3$. *Then there exists a minimum embedding of an* $E_2(u,1)$ *into an* $S_\lambda(2,4,u+w)$ *if and only if the conditions in Table 1 are satisfied.*

Appendix

In this appendix we list some minimum embeddings of an $S_3(2,4,u)(U, \mathcal{C})$ into an $S_\lambda(2,4,u+w)(V, \mathcal{B})$, $V = U \cup W$, for small values of u. *Only* for $\lambda = 3$ we list five minimum embeddings of an $E_2(u,1)$ into an $S_3(2,4,u)$. In these cases we list the blocks of an $E_2(u,1)$-design using square brackets (braces). For example, $[x, y; z, t]$ is the block of an $E_2(u,1)$-design having vertices x, y, z, t and edges $\{x, y\}$ and $\{z, t\}$.

1. $\lambda = 3, u = 4, w = 0$. Let $U = \mathbb{Z}_4$. Blocks: $[0, 1; 2, 3]$, $[0, 3; 1, 2]$, $[0, 2; 1, 3]$.
2. $\lambda = 3, u = 5, w = 0$. Let $U = \mathbb{Z}_5$. Develop (mod 5) the base block $[0, 1; 2, 4]$.
3. $\lambda = 3, u = 8, w = 0$. Let $U = \mathbb{Z}_7 \cup \{\infty\}$. Develop (mod 7) the base blocks $[0, 1; 3, 6]$, $[\infty, 3; 0, 2]$.
4. $\lambda = 3, u = 9, w = 0$. Let $U = \mathbb{Z}_8 \cup \{\infty\}$. Develop (mod 8) the base blocks: $[0, 1; 4, 7]$, $[\infty, 3; 0, 2]$. Add the following blocks: $[0, 4; 2, 6]$, $[1, 5; 3, 7]$.
5. $\lambda = 3, u = 12, w = 0$. Let $U = \mathbb{Z}_{11} \cup \{\infty\}$. Develop (mod 11) the base blocks $[0, 1; 4, 10]$, $[\infty, 6; 0, 4]$, $[0, 3; 4, 6]$.
6. $\lambda = 4, u = 5, w = 11$. Let $V = \mathbb{Z}_5 \cup \{a_i : i \in Z_{11}\}$. Embed an $S_3(2,4,5)$ on $\mathbb{Z}_5$ into an $S_3(2,4,16)$ on V. Paste an $S(2,4,16)$ on V. The result is an $S_4(2,4,16)$ on V which embeds an $S_3(2,4,5)$ on $\mathbb{Z}_5$.
7. $\lambda = 4, u = 17, w = 2$. Let $U = \{i, i' : i \in \mathbb{Z}_8\} \cup \{\infty\}$ and $V = U \cup \{a, b\}$. Take on U an $S_3(2,4,17)$. Put

$$\begin{aligned} C = \{&\{\infty, 1, 7\}, \{\infty, 2, 0\}, \{\infty, 3, 5\}, \{\infty, 4, 6\}, \{1, 1', 5'\}, \{2, 2', 6'\}, \\ &\{3, 3', 7'\}, \{4, 4', 0'\}, \{1, 2', 7'\}, \{2, 3', 0'\}, \{3, 4', 5'\}, \{4, 1', 6'\}, \{5, 2', 3'\}, \\ &\{6, 3', 4'\}, \{7, 4', 1'\}, \{0, 1', 2'\}, \{5, 5', 6'\}, \{6, 6', 7'\}, \{7, 7', 0'\}, \{0, 0', 5'\}\}. \end{aligned}$$

$$\begin{aligned} D = \{&\{\infty, 1', 3'\}, \{\infty, 2', 4'\}, \{\infty, 5', 7'\}, \{\infty, 6', 0'\}, \{1, 4', 6'\}, \{2, 1', 7'\}, \\ &\{3, 2', 0'\}, \{4, 3', 5'\}, \{1, 3', 0\}, \{2, 4', 5\}, \{3, 1', 6\}, \{4, 2', 7\}, \{1, 0', 6\}, \{2, 5', 7\}, \{3, 6', 0\}, \\ &\{4, 7', 5\}, \{5, 1', 0'\}, \{6, 2', 5'\}, \{7, 3', 6'\}, \{0, 4', 7'\}\}. \end{aligned}$$

   Take the blocks $\{a, x, y, z\}$ for any $\{x, y, z\} \in C$ and $\{b, x, y, z\}$ for any $\{x, y, z\} \in D$. At last add the blocks:

$$\{1, 2, 3, 4\}, \{5, 6, 7, 0\}, \{a, b, 1, 5\}, \{a, b, 2, 6\}, \{a, b, 3, 7\}, \{a, b, 4, 0\}.$$

   The result is an $S_4(2,4,19)$ on V which embeds an $S_3(2,4,17)$ on U.

8. $\lambda = 4, u = 20, w = 2$. Let $U = \{i, i' : i \in \mathbb{Z}_{10}\}$ and $V = U \cup \{a, b\}$. Take on U an $S_3(2, 4, 20)$. Put

$$C = \{\{0', 2, 4\}, \{1', 1, 3\}, \{2', 6, 8\}, \{3', 5, 7\}, \{4', 6, 3\}, \{5', 1, 8\}, \{6', 4, 5\}, \{7', 7, 2\}, \{9, 1, 6\}, \{9, 2, 8\}, \{9, 3, 5\}, \{9, 4, 7\}, \{0, 0', 5'\}, \{0, 1', 6'\}, \{0, 2', 7'\}, \{0, 3', 4'\}, \{8', 4, 4'\}, \{8', 2, 2'\}, \{8', 5, 5'\}, \{8', 7, 1'\}, \{9', 0', 3\}, \{9', 3', 6\}, \{9', 6', 8\}, \{9', 7', 1\}\}.$$

$$D = \{\{0', 6, 7\}, \{1', 5, 8\}, \{2', 1, 4\}, \{3', 2, 3\}, \{4', 7, 8\}, \{5', 3, 4\}, \{6', 1, 2\}, \{7', 5, 6\}, \{9, 0', 6'\}, \{9, 1', 7'\}, \{9, 2', 4'\}, \{9, 3', 5'\}, \{0, 1, 7\}, \{0, 2, 5\}, \{0, 3, 8\}, \{0, 4, 6\}, \{8', 3', 1\}, \{8', 7', 3\}, \{8', 0', 8\}, \{8', 6', 6\}, \{9', 2, 4'\}, \{9', 5, 2'\}, \{9', 4, 1'\}, \{9', 7, 5'\}\}.$$

Take the blocks $\{a, x, y, z\}$ for any $\{x, y, z\} \in C$ and $\{b, x, y, z\}$ for any $\{x, y, z\} \in D$. At last add the blocks:

$$\{9, 0, 8', 9'\}, \{0', 1', 2', 3'\}, \{4', 5', 6', 7'\}, \{0', 4', 1, 5\}, \{1', 5', 2, 6\}, \{2', 6', 3, 7\}, \{3', 7', 4, 8\}, \{a, b, 0', 7'\}, \{a, b, 1', 4'\}, \{a, b, 2', 5'\}, \{a, b, 3', 6'\}.$$

The result is an $S_4(2, 4, 22)$ on V which embeds an $S_3(2, 4, 20)$ on U.

9. $\lambda = 4, u = 8, w = 14$. Let $U = \{a_i : i \in \mathbb{Z}_8\}$, $W = \mathbb{Z}_{14}$ and $V = U \cup W$. Take on U an $S_3(2, 4, 8)$. The edges of K_{14} may be factored into a set of 7 disjoint classes $P_1, P_2, \ldots, P_7$ where $(i, j) \in P_k$ if and only of $i - j \equiv k \pmod{14}$. For $i \in Z_{14}$, let $T_0 = \{i, 6 + i, 5 + i\}$, $T_1 = \{i, 2 + i, 5 + i\}$, $T_2 = \{i, 2 + i, 6 + i\}$, $T_3 = \{i, 3 + i, 4 + i\}$ be four sets of 14 triangles covering, respectively, $P_1, P_2, \ldots, P_6$ repeated twice. For $i = 0, 1, 2, 3$, put $T_{i+4} = T_i$. For $i = 0, 1, \ldots, 7$, construct the blocks $\{a_i, x, y, z\}$, $\{x, y, z\} \in T_i$. Let $F_0, F_2, \ldots, F_6$ be the 1-factors of a 1-factorization of the complete graph K_8 on U. For $i = 0, 1, \ldots, 6$, construct the blocks $\{i, i + 7, x, y\}$, $\{x, y\} \in F_i$. The result is an $S_4(2, 4, 22)$ on V which embeds an $S_3(2, 4, 8)$ on U.

10. $\lambda = 5, u = 5, w = 8$. Let $U = \{a_i : i \in \mathbb{Z}_5\}$, $W = \mathbb{Z}_8$ and $V = U \cup W$. Take on U an $S_3(2, 4, 5)$. For $i \in \mathbb{Z}_5$, develop (mod 8) the base block $\{a_i, 0, 1, 3\}$. Add the blocks:

$$\{a_0, a_1, 0, 4\}, \{a_0, a_1, 1, 5\}, \{a_0, a_2, 2, 6\}, \{a_0, a_2, 3, 7\}, \{a_0, a_4, 0, 4\}, \{a_0, a_4, 1, 5\}, \{a_0, a_3, 2, 6\}, \{a_0, a_3, 3, 7\}, \{a_1, a_3, 0, 4\}, \{a_1, a_3, 1, 5\}\{a_1, a_2, 2, 6\}, \{a_1, a_2, 3, 7\}, \{a_2, a_3, 0, 4\}, \{a_2, a_3, 1, 5\}, \{a_1, a_4, 2, 6\}\{a_1, a_4, 3, 7\}, \{a_2, a_4, 0, 4\}, \{a_2, a_4, 1, 5\}, \{a_3, a_4, 2, 6\}, \{a_3, a_4, 3, 7\}.$$

The result is an $S_5(2, 4, 13)$ on V which embeds an $S_3(2, 4, 5)$ on U.

11. $\lambda = 5, u = 8, w = 5$. Let $U = \mathbb{Z}_8$, $W = \{a, b, c, d, e\}$ and $V = U \cup W$. Take on U an $S_3(2, 4, 8)$. Add the blocks:

$$\{a, b, c, 1\}, \{a, 1, 2, 3\}, \{b, 1, 4, 5\}, \{c, 6, 7, 8\}, \{a, b, 1, 4\}, \{a, b, 2, 6\}, \{a, b, 3, 8\}, \{a, b, 5, 7\}, \{a, c, 1, 6\}, \{a, c, 2, 3\}, \{a, c, 4, 8\}, \{a, c, 5, 7\}, \{b, c, 1, 7\}, \{b, c, 2, 6\}, \{b, c, 3, 4\}, \{b, c, 5, 8\}, \{a, d, 2, 4\}, \{a, d, 3, 5\}, \{a, d, 4, 8\}, \{a, d, 2, 5\}, \{a, d, 6, 7\}, \{a, e, 1, 8\}, \{a, e, 6, 8\}, \{a, e, 5, 6\}, \{a, e, 3, 7\}, \{a, e, 4, 7\}, \{b, d, 1, 6\}, \{b, d, 3, 6\}, \{b, d, 4, 7\}, \{b, d, 3, 8\}, \{b, d, 8, 5\}, \{b, e, 6, 4\}, \{b, e, 2, 7\}, \{b, e, 3, 7\}, \{b, e, 2, 8\}, \{b, e, 2, 5\}, \{c, d, 1, 2\}, \{c, d, 4, 6\}, \{c, d, 2, 7\}, \{c, d, 3, 5\}, \{c, d, 7, 8\}, \{c, e, 1, 3\}, \{c, e, 4, 5\}, \{c, e, 5, 6\}, \{c, e, 3, 4\}, \{c, e, 2, 8\}, \{d, e, 1, 5\}, \{d, e, 1, 7\}, \{d, e, 1, 8\}, \{d, e, 2, 4\}, \{d, e, 3, 6\}.$$

The result is an $S_5(2, 4, 13)$ on V which embeds an $S_3(2, 4, 8)$ on U.

12. $\lambda = 5, u = 21, w = 4$. Let $U = \mathbb{Z}_{21}$ and $V = \mathbb{Z}_5 \cup \{a_0, a_1, a_2, a_3\}$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 21)$ on $\mathbb{Z}_{21}$. Take on U a resolvable $S_2(2, 3, 21)$ having the resolution classes R_j, $j = 0, 1, \ldots, 19$. For each $i = 0, 1, 2, 3$, place the blocks $\{a_i, x, y, z\}$, $\{x, y, z\} \in \bigcup_{j=0}^{4} \mathcal{R}_{5i+j}$. Add the blocks of an $S_5(2, 4, 4)$ on $\{a_0, a_1, a_2, a_3\}$ and the result is an $S_5(2, 4, 25)$ on V which embeds an $S_3(2, 4, 21)$ on U.

13. $\lambda = 5, u = 17, w = 8$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 17)$ having $U = (\mathbb{Z}_8 \times \{0, 1\}) \cup \{\infty\}$ as point-set and let V be the set $(\mathbb{Z}_8 \times \{0, 1, 2\}) \cup \{\infty\}$.

Let us develop (mod 8) the following 24 base blocks:

$$\{0_0, 2_0, 4_0, 6_0\}, \{0_0, 4_0, 1_2, 5_2\}, \{0_0, 3_0, 7_2, \infty\}, \{0_1, 2_2, 5_2, \infty\}, \{0_1, 3_2, 5_2, \infty\}, \{0_0, 0_1, 0_2, 1_2\}, \{0_0, 0_1, 0_2, 2_2\}, \{0_0, 1_1, 0_2, 1_2\}, \{0_0, 1_1, 0_2, 2_2\}, \{0_0, 2_1, 0_2, 1_2\}, \{0_0, 2_1, 1_2, 3_2\}, \{0_0, 3_1, 2_2, 6_2\}, \{0_0, 3_1, 4_2, 7_2\}, \{0_0, 4_1, 4_2, 7_2\}, \{0_0, 4_1, 6_2, 7_2\}, \{0_0, 5_1, 3_2, 7_2\}, \{0_0, 6_1, 2_2, 4_2\}, \{0_0, 6_1, 3_2, 6_2\}, \{0_0, 7_1, 2_2, 5_2\}, \{0_1, 1_1, 3_1, 5_2\}, \{0_1, 1_1, 4_1, 5_2\}, \{0_0, 1_0, 3_0, 6_2\}, \{0_0, 1_0, 5_2, 6_2\}, \{0_0, 5_1, 7_1, 3_2\}.$$

We get 2 blocks from the first base block, 4 blocks from the second base block, and 8 blocks from each one of the other twenty-two base blocks. Add these 182 blocks to the 68 of $\mathcal{C}$ and denote by $\mathcal{B}$ the set containing all these 250 blocks. The result is an $S_5(2, 4, 25)$ $(V, \mathcal{B})$ which embeds an $S_3(2, 4, 17)$ $(U, \mathcal{C})$.

14. $\lambda = 5, u = 20, w = 5$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 20)$ having $U = \mathbb{Z}_5 \times \{0, 1, 2, 3\}$ as point-set and let V be the set $\mathbb{Z}_5 \times \{0, 1, 2, 3, 4\}$. Let us develop (mod 5) the following 31 base blocks: $\{0_3, 0_2, 0_1, 4_0\}$,

$$\{0_4, 0_3, 1_3, 2_3\}, \{0_4, 0_2, 1_2, 2_2\}, \{0_4, 0_1, 1_1, 2_1\}, \{0_4, 0_0, 1_0, 2_0\}, \{0_4, 1_4, 0_3, 2_3\},$$
$$\{0_4, 1_4, 0_2, 2_2\}, \{0_4, 1_4, 0_1, 2_1\}, \{0_4, 1_4, 0_0, 2_0\}, \{0_4, 1_4, 0_3, 0_2\}, \{0_4, 2_4, 0_3, 0_1\},$$
$$\{0_4, 2_4, 0_3, 0_0\}, \{0_4, 2_4, 0_2, 0_1\}, \{0_4, 2_4, 0_2, 0_0\}, \{0_4, 2_4, 0_1, 0_0\}, \{0_4, 1_3, 1_2, 1_1\},$$
$$\{0_4, 1_3, 2_2, 3_1\}, \{0_4, 1_3, 2_2, 3_1\}, \{0_4, 2_3, 4_2, 1_1\}, \{0_4, 2_3, 1_2, 3_0\}, \{0_4, 2_3, 3_2, 4_0\},$$
$$\{0_4, 3_3, 3_2, 4_0\}, \{0_4, 3_3, 3_2, 1_0\}, \{0_4, 4_3, 4_1, 1_0\}, \{0_4, 4_3, 4_1, 2_0\}, \{0_4, 4_3, 4_1, 4_0\},$$
$$\{0_4, 3_3, 1_1, 2_0\}, \{0_4, 1_2, 2_1, 4_0\}, \{0_4, 2_2, 4_1, 2_0\}, \{0_4, 4_2, 2_1, 1_0\}, \{0_4, 4_2, 2_1, 3_0\}.$$

We get 5 blocks from each base block. Add these 155 blocks to the 95 of $\mathcal{C}$ and denote by $\mathcal{B}$ the set containing all these 250 blocks. The result is an $S_5(2, 4, 25)$ $(V, \mathcal{B})$ which embeds an $S_3(2, 4, 20)$ $(U, \mathcal{C})$.

15. $\lambda = 5, u = 41, w = 8$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 41)$ having $U = (\mathbb{Z}_8 \times \{0, 1, 2, 3, 4\}) \cup \{\infty\}$ as point-set and let V be the set $(\mathbb{Z}_8 \times \{0, 1, 2, 3, 4, 5\}) \cup \{\infty\}$. Let us develop (mod 8) the following 72 base blocks: $\{0_5, 2_5, 4_5, 6_5\}$, $\{0_3, 1_2, 1_1, 4_0\}$,

$$\{0_5, 0_4, 1_4, 2_4\}, \{0_5, 0_4, 2_4, 5_4\}, \{0_5, 0_3, 1_3, 2_3\}, \{0_5, 0_3, 2_3, 5_3\}, \{0_5, 0_2, 1_2, 2_2\}, \{0_5, 0_2, 2_2, 5_2\},$$
$$\{0_5, 0_1, 1_1, 2_1\}, \{0_5, 0_1, 2_1, 5_1\}, \{0_5, 0_0, 1_0, 2_0\}, \{0_5, 0_0, 2_0, 5_0\}, \{0_5, 1_5, 0_4, 4_4\},$$
$$\{0_5, 1_5, 0_3, 4_3\}, \{0_5, 1_5, 0_2, 4_2\}, \{0_5, 1_5, 0_1, 4_1\}, \{0_5, 1_5, 0_0, 4_0\}, \{0_5, 2_5, 0_4, 0_3\},$$
$$\{0_5, 2_5, 1_4, 0_2\}, \{0_5, 2_5, 1_4, 1_1\}, \{0_5, 2_5, 1_4, 1_0\}, \{0_5, 3_5, 1_3, 1_2\}, \{0_5, 3_5, 1_3, 1_1\},$$
$$\{0_5, 3_5, 1_3, 1_0\}, \{0_5, 3_5, 1_2, 1_1\}, \{0_5, 3_5, 1_2, 1_0\}, \{0_5, 4_5, 1_1, 1_0\}, \{0_5, 1_4, 2_3, 1_2\},$$
$$\{0_5, 2_4, 1_3, 3_2\}, \{0_5, 2_4, 3_3, 4_2\}, \{0_5, 2_4, 4_3, 3_2\}, \{0_5, 3_4, 2_3, 3_1\}, \{0_5, 3_4, 5_3, 2_1\},$$
$$\{0_5, 3_4, 6_3, 2_1\}, \{0_5, 3_4, 7_3, 4_1\}, \{0_5, 4_4, 2_3, 2_0\}, \{0_5, 4_4, 7_3, 2_0\}, \{0_5, 5_4, 3_3, 2_0\},$$
$$\{0_5, 4_4, 2_2, 5_1\}, \{0_5, 4_4, 2_2, 6_1\}, \{0_5, 5_4, 2_2, 3_1\}, \{0_5, 5_4, 4_2, 2_1\}, \{0_5, 5_4, 7_2, 4_0\},$$
$$\{0_5, 6_4, 3_2, 5_0\}, \{0_5, 6_4, 7_2, 3_0\}, \{0_5, 6_4, 3_1, 7_0\}, \{0_5, 6_4, 4_1, 7_0\}, \{0_5, 7_4, 7_1, 3_0\},$$
$$\{0_5, 3_3, 3_2, 5_1\}, \{0_5, 4_3, 4_2, 5_1\}, \{0_5, 5_3, 5_2, 7_1\}, \{0_5, 4_3, 5_2, 7_0\}, \{0_5, 7_3, 7_2, 4_0\},$$
$$\{0_5, 4_3, 5_2, 6_0\}, \{0_5, 7_3, 6_2, 4_0\}, \{0_5, 3_3, 7_1, 5_0\}, \{0_5, 5_3, 4_1, 6_0\}, \{0_5, 5_3, 6_1, 6_0\},$$
$$\{0_5, 5_2, 4_1, 3_0\}, \{0_5, 4_2, 3_1, 5_0\}, \{0_5, 7_2, 6_1, 3_0\}, \{0_4, 5_3, 3_2, 3_0\}, \{0_4, 5_3, 3_1, 4_0\},$$
$$\{0_4, 4_3, 3_1, 2_0\}, \{0_4, 3_2, 4_1, 2_0\}, \{0_4, 4_2, 2_1, 3_0\}, \{0_3, 1_2, 7_1, 4_0\}, \{0_5, 0_4, 0_0, \infty\},$$
$$\{0_5, 0_2, 0_1, \infty\}, \{0_5, 0_3, 0_1, \infty\}, \{0_4, 0_3, 0_2, \infty\}, \{0_5, 4_5, 0_0, \infty\}.$$

We get 2 blocks from the first base block and 8 blocks from each one of the seventy-one other base blocks. Add these 570 blocks to the 410 of $\mathcal{C}$ and denote by $\mathcal{B}$ the set containing all these 980 blocks. The result is an $S_5(2, 4, 49)$ $(V, \mathcal{B})$ which embeds an $S_3(2, 4, 41)$ $(U, \mathcal{C})$.

16. $\lambda = 6, u = 5, w = 1$. Let $U = \mathbb{Z}_5$ and $V = \mathbb{Z}_5 \cup \{\infty\}$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 5)$ on $\mathbb{Z}_5$. Develop (mod 5) the base blocks: $\{\infty, 0, 1, 2\}, \{\infty, 0, 2, 3\}$. The result is an $S_6(2, 4, 6)$ on V which embeds an $S_3(2, 4, 5)$ on $\mathbb{Z}_5$.
17. $\lambda = 6, u = 5, w = 5$. Let $U = \mathbb{Z}_5 \times \{0\}$, $W = \mathbb{Z}_5 \times \{1\}$ and $V = U \cup W$. Let $(U, \mathcal{C})$ be an $S_3(2, 4, 5)$ on U. Develop (mod 5) the base blocks: $\{0_0, 1_0, 0_1, 1_1\}, \{0_0, 1_0, 2_1, 3_1\}, \{0_0, 1_0, 3_1, 4_1\}, \{0_0, 2_0, 0_1, 2_1\}, \{0_0, 2_0, 4_1, 1_1\}, \{0_0, 2_0, 1_1, 3_1\}$, $\{0_0, 0_1, 2_1, 3_1\}, \{0_0, 0_1, 1_1, 4_1\}$. The result is an $S_6(2, 4, 10)$ on V which embeds the given $S_3(2, 4, 5)$ on U.
18. $\lambda = 6, u = 8, w = 1$. Let $U = \mathbb{Z}_8$ and $V = \mathbb{Z}_8 \cup \{\infty\}$. Take on U an $S_3(2, 4, 8)$ and add the blocks $\{\infty, 0, 1, 4\}, \{\infty, 1, 2, 5\}, \{\infty, 2, 3, 6\}, \{\infty, 3, 0, 7\}, \{\infty, 4, 5, 0\}, \{\infty, 5, 6, 1\}, \{\infty, 6, 7, 2\}, \{\infty, 7, 4, 3\}, \{\infty, 0, 1, 6\}$, $\{\infty, 1, 2, 7\}, \{\infty, 2, 3, 4\}, \{\infty, 3, 0, 5\}, \{\infty, 4, 5, 2\}, \{\infty, 5, 6, 3\}, \{\infty, 6, 7, 0\}, \{\infty, 7, 4, 1\}, \{0, 1, 2, 3\}, \{4, 5, 6, 7\}$ $\{0, 2, 4, 6\}, \{1, 3, 5, 7\}, \{0, 2, 5, 7\}, \{1, 3, 4, 6\}$. The result is an $S_6(2, 4, 9)$ on V which embeds an $S_3(2, 4, 8)$ on U.
19. $\lambda = 6, u = 17, w = 1$. Let $U = \mathbb{Z}_{17}$ and $V = \mathbb{Z}_{17} \cup \{\infty\}$. Take an $S_3(2, 4, 17)$ on U. Develop (mod 17) the base blocks: $\{\infty, 0, 6, 7\}, \{\infty, 0, 2, 7\}, \{0, 4, 6, 9\}, \{0, 1, 3, 12\}, \{0, 4, 7, 8\}$. The result is an $S_6(2, 4, 18)$ on V which embeds an $S_3(2, 4, 17)$ on $\mathbb{Z}_{17}$.
20. $\lambda = 6, u = 8, w = 2$. Let $U = \mathbb{Z}_8$ and $V = \mathbb{Z}_8 \cup \{a, b\}$. Take on U an $S_3(2, 4, 8)$ and add the blocks

$$\{a, b, 0, 2\}, \{a, b, 0, 2\}, \{a, b, 1, 3\}, \{a, b, 1, 3\}, \{a, b, 4, 6\}, \{a, b, 5, 7\},$$
$$\{a, 0, 1, 4\}, \{a, 1, 2, 5\}, \{a, 2, 3, 6\}, \{a, 3, 0, 7\}, \{a, 0, 4, 5\}, \{a, 0, 6, 7\},$$
$$\{a, 1, 4, 7\}, \{a, 1, 6, 5\}, \{a, 2, 5, 7\}, \{a, 2, 4, 6\}, \{a, 3, 4, 5\}, \{a, 3, 6, 7\},$$
$$\{b, 0, 1, 6\}, \{b, 1, 2, 7\}, \{b, 2, 3, 4\}, \{b, 3, 0, 5\}, \{b, 0, 5, 6\}, \{b, 0, 4, 7\},$$
$$\{b, 1, 4, 6\}, \{b, 1, 5, 7\}, \{b, 2, 4, 5\}, \{b, 2, 6, 7\}, \{b, 3, 4, 7\}, \{b, 3, 5, 6\}, \{0, 1, 2, 3\}.$$

The result is an $S_6(2, 4, 10)$ on V which embeds an $S_3(2, 4, 8)$ on U.

21. $\lambda = 6, u = 29, w = 7$. Let $U = \mathbb{Z}_{29}$, $W = \{a_i : i \in Z_7\}$ and $V = U \cup W$. Take an $S_3(2, 4, 29)$ and develop (mod 29) the base blocks: $\{a_0, 0, 1, 3\}, \{a_0, 0, 4, 11\}, \{a_1, 0, 10, 24\}, \{a_1, 0, 11, 12\}, \{a_2, 0, 5, 7\}, \{a_2, 0, 5, 8\}, \{a_3, 0, 10, 23\}, \{a_3, 0, 12, 25\}, \{a_4, 0, 8, 14\}, \{a_4, 0, 9, 18\}, \{a_5, 0, 12, 13\}, \{a_5, 0, 8, 10\}, \{a_6, 0, 14, 20\}, \{a_6, 0, 4, 7\}$. Add the blocks of an $S_6(2, 4, 7)$ on W. The result is an $S_6(2, 4, 36)$ on V which embeds an $S_3(2, 4, 29)$ on U.

22. $\lambda = 7, u = 9, w = 4$. Let $U = \mathbb{Z}_3 \times \{0, 1, 2\}$, $W = \{a_i : i \in \mathbb{Z}_4\}$ and $V = U \cup W$. Take on U an $S_3(2, 4, 9)$. Develop (mod 3) the base blocks:

$$\{a_1, a_2, 0_0, 1_0\}, \{a_1, a_2, 0_1, 0_2\}, \{a_1, a_3, 0_1, 1_1\}, \{a_1, a_3, 0_0, 0_2\}, \{a_1, a_0, 0_2, 1_2\}, \{a_1, a_0, 0_0, 0_1\},$$
$$\{a_2, a_3, 0_0, 1_2\}, \{a_2, a_3, 0_1, 1_2\}, \{a_2, a_0, 0_0, 1_1\}, \{a_2, a_0, 0_1, 2_2\}, \{a_3, a_0, 0_0, 2_1\}, \{a_3, a_0, 0_0, 2_2\}.$$

Take on U a resolvable $S_2(2, 3, 9)$ having the resolution classes R_j, $j = 0, 1, \ldots, 7$. For each $i = 0, 1, 2, 3$, place the blocks $\{a_i, x, y, z\}, \{x, y, z\} \in \bigcup_{j=0}^{1} \mathcal{R}_{2i+j}$. Add the blocks of an $S(2, 4, 13)$ on V and the result is an $S_7(2, 4, 13)$ on V which embeds an $S_3(2, 4, 9)$ on U.
23. $\lambda = 7, u = 29, w = 8$. Let $U = \mathbb{Z}_{29}$, $W = \{a_i : i \in \mathbb{Z}_8\}$ and $V = U \cup W$. Take on $U \cup \{a_i : i \in \mathbb{Z}_7\}$ an $S_6(2, 4, 36)$ which embeds an $S_3(2, 4, 29)$ on U (see case 21 in the Appendix). Construct on $U \cup \{a_i : i \in Z_7\} \cup \{\infty_i : i \in \mathbb{Z}_7\}$ a 4-GDD of type $7^1 1^{36}$ having $\{\infty_i : i \in \mathbb{Z}_7\}$ as a group of size 7. Replace, for each $i \in \mathbb{Z}_7$, ∞_i with a_7 and take the blocks so obtained. The result is an $S_7(2, 4, 37)$ on V which embeds an $S_3(2, 4, 29)$ on U.
24. $\lambda = 7, u = 32, w = 5$. Let $U = \mathbb{Z}_{31} \cup \{\infty\}$, $W = \{a_i : i \in \mathbb{Z}_5\}$ and $V = U \cup W$. Take an $S_3(2, 4, 32)$ on U. Develop (mod 31) the base blocks: $\{\infty, 0, 11, 12\}$, $\{a_0, 0, 7, 9\}$, $\{a_0, 0, 5, 8\}$, $\{a_1, 0, 13, 27\}$, $\{a_1, 0, 14, 16\}$, $\{a_2, 0, 13, 20\}$, $\{a_2, 0, 6, 14\}$, $\{a_3, 0, 3, 10\}$, $\{a_3, 0, 12, 13\}$, $\{a_4, 0, 1, 3\}$, $\{a_4, 0, 12, 20\}$, $\{0, 5, 9, 15\}$, $\{0, 5, 9, 15\}$. Add the blocks of an $S(2, 4, 37)$ on V. The result is an $S_7(2, 4, 37)$ on V which embeds an $S_3(2, 4, 32)$ on U.
25. $\lambda = 7, u = 53, w = 8$. Let $U = \mathbb{Z}_{48} \cup \{b_i : i \in \mathbb{Z}_5\}$, $W = \{a_i : i \in \mathbb{Z}_8\}$ and $V = U \cup W$. Construct an $S_3(2, 4, 53)$ on U. Give weight 7 to every point of W and weight 4 to every point of $\{b_i : i \in \mathbb{Z}_5\}$, construct on V four 4-GDD of type $19^1 1^{48}$. On $\{a_i, i \in \mathbb{Z}_8\} \cup \{b_i : i \in \mathbb{Z}_5\}$, place an $S_7(2, 4, 13)(V, \mathcal{B})$ which embeds an $S_3(2, 4, 5)(Y, \mathcal{C})$ on $\{b_i : i \in Z_5\}$. Delete the blocks of $\mathcal{C}$ and take the blocks so obtained. The result is an $S_7(2, 4, 61)$ on V which embeds an $S_3(2, 4, 53)$ on U.
26. $\lambda = 7, u = 44, w = 5$. Let $U = \mathbb{Z}_{39} \cup \{b_i : i \in \mathbb{Z}_5\}$, $W = \{a_i : i \in \mathbb{Z}_5\}$ and $V = U \cup W$. Take an $S_3(2, 4, 44)$ on U. Give weight 6 to every point of W and weight 3 to every point of $Y = \{b_i : i \in \mathbb{Z}_5\}$, construct on V two 4-GDD of type $19^1 1^{39}$ and a 4-GDD of type $7^1 1^{39}$. On $\{a_i : i \in \mathbb{Z}_5\} \cup \{b_i : i \in \mathbb{Z}_5\}$, place an $S_6(2, 4, 10)(X, \mathcal{B})$ which embeds an $S_3(2, 4, 5)(Y, \mathcal{C})$ on $\{b_i : i \in Z_5\}$ (see case 17). Delete the blocks of $\mathcal{C}$ and take the blocks so obtained. Finally paste an $S(2, 4, 49)$ on V. The result is an $S_7(2, 4, 49)$ on V which embeds an $S_3(2, 4, 44)$ on U.
27. $\lambda = 8, u = 8, w = 2$. The result follows by pasting an $S_2(2, 4, 10)$ to an $S_6(2, 4, 10)$ which embeds an $S_3(2, 4, 8)$ on U (see case 20).
28. $\lambda = 10, u = 5, w = 2$. Let $U = \mathbb{Z}_5$ and $V = \mathbb{Z}_5 \cup \{\infty_1, \infty\}$. Construct on $\mathbb{Z}_5 \cup \{\infty_1\}$ an $S_6(2, 4, 6)$ which embeds an $S_3(2, 4, 5)$ on $\mathbb{Z}_5$ (see case 16). Take on $\mathbb{Z}_5 \cup \{\infty_1\}$ an $S_4(2, 3, 6)$ having block set $\mathcal{B}$ and form the blocks $\{\infty, x, y, z\}$, for each $\{x, y, z\} \in \mathcal{B}$. The result is an $S_{10}(2, 4, 7)$ on V which embeds an $S_3(2, 4, 5)$ on U.

References

[1] A.E. Brouwer, Optimal packings of K_4's into a K_n, J. Comb. Theory Ser. A 26 (1979) 278–297.
[2] C.J. Colbourn, J.H. Dinitz, The CRC Handbook of Combinatorial Designs, Chapman and Hall, CRC, Boca Raton, FL, 2007, Online updates www.emba.uvm.edu/~dinitz/newresults.html.
[3] C.J. Colbourn, A.C.H. Ling, G. Quattrocchi, Minimum embedding of P_3-designs into $(K_4 - e)$-designs, J. Combin. Des. 11 (2003) 352–366.
[4] P. Danziger, S. Milici, G. Quattrocchi, Minimum embedding of a P_4-design into a balanced block design of index λ, Discrete Math. 309 (2009) 4861–4870.
[5] M. Gionfriddo, S. Milici, Minimum embedding of a K_3-design into a balanced incomplete block design of index $\lambda \geq 2$. Ars Combin. (in press).
[6] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975) 255–369.
[7] S. Milici, Minimum embedding of P_3-designs into $TS(v, \lambda)$, Discrete Math. 308 (2008) 331–338.
[8] S. Milici, G. Ragusa, Maximum embedding of an $H(v - w, 3, 1)$ into a $TS(v, \lambda)$, Australas. J. Combin. 46 (2010) 121–127.
[9] S. Milici, F. Zuanni, On the existence of an $S_{(\lambda)}(2; 4; v)$ which embeds an $S(2; 3; v - w)$ of maximum order for $\lambda \geq 2$, BICA 60 (2010) 69–78.
[10] G. Quattrocchi, Embedding G_1-designs into G_2-designs, a short survey, in: 6th Workshop on Combinatorics. Rend. Sem. Mat. Messina, Ser II Suppl., vol. 8, 2002, pp. 129–143.